Differential Equations

Free Response Questions

Q: 1 **Sumit finds the general solution of the differential equation $yy' = e^{3x}$ as $y = \frac{1}{3}e^{3x} + C$, where C is the arbitrary constant.** **[1]**

Did Sumit find the correct general solution? Show your work and justify.

Q: 2 **The bottom valve of a conical tank is opened to remove sugarcane juice in a factory. The rate at which the juice pours out from the conical tank is directly proportional to the cube root of the rate of change of height of the juice present in the tank.** **[1]**

If k is the constant of proportionality, write a differential equation depicting the scenario.

Q: 3 **A differential equation is given below.** **[1]**

$$\left(\frac{d^3y}{dx^3}\right)^3 + \left(\frac{d^4y}{dx^4}\right)^2 - \left(\frac{dy}{dx}\right)^4 + 7y = 21$$

State whether the following statement is true or false. Justify your answer.

The degree of the given differential equation is equal to its order.

Q: 4 **[1]**

$$\left(\frac{d^4y}{dx^4}\right)^5 + \left(\frac{d^3y}{dx^3}\right)^6 - \left(\frac{d^2y}{dx^2}\right) + \left(\frac{dy}{dx}\right)^2 = 25$$

A differential equation is given above. State whether the following statement is true or false. Justify your answer.

The general solution of the given differential equation will have five arbitrary constants.

Q: 5 **If the tangent to a curve at every single point on it is given by $y - \frac{2y}{x+1}$ then find the equation of the curve. Show your steps.** **[2]**

Q: 6 **In a controlled condition within a laboratory, a spherical balloon is being deflated at a rate proportional to its surface area at that instant. The spherical shape of the balloon is maintained throughout the process.** **[2]**

Form a differential equation that represents the rate of change of its radius. Show your work.

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Q: 7 **What family of curves does the solution of the differential equation $2x\frac{dy}{dx} = 7 + y$ represent? Show your work.** **[2]**

Q: 8 **The solution of the differential equation $xdy = (3x - 2y)\,dx$ is in the form:** **[3]**

$x^3 = C.\,f(x, y)$, where C is an arbitrary constant.

Find $f(x, y)$. Show your work.

Q: 9 **Birds are sensitive to microwaves that are emitted by mobile phones, which has resulted in a decline in the bird population, especially sparrows.** **[5]**

The population of sparrows in a certain region is decreasing according to the following equation due to the extensive use of mobile phones.

$\frac{dy}{dt} = ky$

where, y represents the population of sparrows at time t (in years) and k is a constant.

The population, which was e^{10} five years ago, has decreased by 25% in that time. Find k. Show your steps.

(Note: Take ln 3 ≈ 1.09 and ln 4 ≈ 1.38.)

Q: 10 **i) Find the differential equation representing the family of curves $(y - c)^2 = x^3$, where c is arbitrary constant.** **[5]**

ii) Find the order of the differential equation found in part i).

iii) Find the degree of the differential equation found in part i) if defined.

Show your work.

Case Study

Answer the questions based on the given information.

The mixing tank shown below generates saline water (a mixture of salt and water) for the cooling of a thermoelectric power plant.

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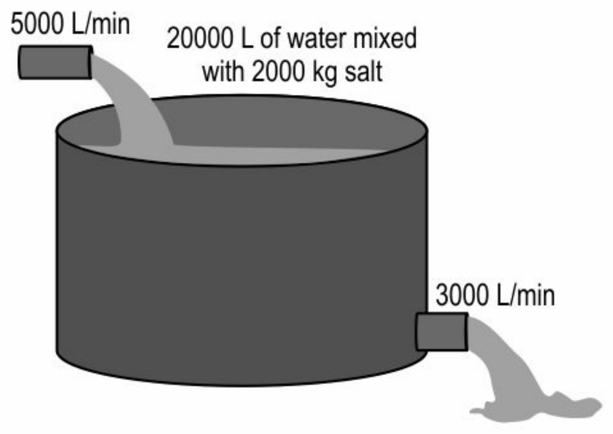

The tank initially holds 20000 L of water in which 2000 kg of salt has been dissolved. Then, pure water is poured into the tank at a rate of 5000 L per minute. The mixture in the tank, which is stirred continuously, flows out at a rate of 3000 L per minute.

The quantity of salt in the tank at time t is denoted by Q_t, where t is in minutes and Q_t is in kilograms.

The rate of flow of salt into the tank is measured as:

$$\left(\frac{dQ_t}{dt}\right)_{in} = 0 \text{ kg/min}$$

The rate of flow of salt out of the tank is measured as:

$$\left(\frac{dQ_t}{dt}\right)_{out} = \text{Rate of flow of water out of the tank} \times \frac{\text{Quantity of salt in the tank at time } t}{\text{Amount of water in the tank at time } t}$$

The rate of change of quantity of salt in the tank with respect to time is given by

$$\left[\left(\frac{dQ_t}{dt}\right)_{in} - \left(\frac{dQ_t}{dt}\right)_{out}\right]$$

Q: 11 **Find the expression for the rate of change of quantity of salt in the tank with time t. Show your work.** **[1]**

Q: 12 **Find the general solution of the differential equation corresponding to the rate of change of quantity of salt in the tank with time t. Show your work.** **[2]**

Q: 13 **Use the initial conditions to determine the particular solution for the differential equation obtained. Show your work.** **[2]**

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Q.No	What to look for	Marks
1	**Rewrites the given differential equation as $ydy = e^{3x}\,dx$.**	**0.5**
	Concludes that Sumit's general solution is incorrect by integrating both sides of the above equation to find the general solution as: $y^2 = \frac{2}{3} e^{3x} + C$**, where C is the arbitrary constant.**	**0.5**
2	**Writes a differential equation that depicts the given scenario as:** $\frac{dV}{dt} = k\sqrt[3]{\frac{dh}{dt}}$ **where, *V* is the volume, *h* is the height of the juice and *t* is the time.**	**1**
3	**Writes that the statement is false.**	**0.5**
	Gives a reason. For example, the order of the given differential equation is 4, but its degree is 2.	**0.5**
4	**Writes that the statement is false.**	**0.5**
	Gives a reason. For example, the number of arbitrary constants in the general solution of a differential equation is determined by its order, not its degree. Since the order of the given equation is 4, it will have only 4 arbitrary constants.	**0.5**
5	**Writes that** $\frac{dy}{dx} = y - \frac{2y}{x+1} = y\left(1 - \frac{2}{x+1}\right)$. **Separates the variables as follows:** $\frac{dy}{y} = \left(1 - \frac{2}{x+1}\right) dx$	**0.5**
	Integrates the above equation to get: $\log_e y = x - 2\log_e(x+1) + c$ $\Rightarrow \log_e y = x - \log_e(x+1)^2 + c$ $\Rightarrow \log_e y + \log_e(x+1)^2 = x + c$ **where c is a constant of integration.**	**1**

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Q.No	What to look for	Marks
	Solves the above equation to find the equation of the curve as follows: $\log_e (y .(x +1)^2) = x + c$ $\Rightarrow y .(x +1)^2 = e^{x+c}$ $\therefore y = \frac{ke^x}{(x+1)^2}$, where k is a constant	**0.5**
6	**Takes V, S and *r* to be the volume, surface area and radius of the balloon at time *t*.** **Writes that since the volume of the balloon is decreasing with time, the rate of change of its radius is negative.**	**0.5**
	Uses the given information and expresses the relationship between volume and surface area as: $\frac{dV}{dt} = -k \times S$, **where *k* is a positive real number**	**0.5**
	Differentiates the above equation after substituting the expressions for volume and surface area of a sphere to get the required differential equation as: $\frac{4}{3}\pi(3r^2)\frac{dr}{dt} = -k \times 4\pi r^2$ $\Rightarrow \frac{dr}{dt} = -k$	**1**
7	**Integrates the given differential equation as:** $\int \frac{dy}{7+y} = \int \frac{dx}{2x}$ $\Rightarrow \log(7+y) = \frac{1}{2}\log x + \log c$ $\Rightarrow (7+y) = \sqrt{x}\, c$ $\Rightarrow (7+y)^2 = x\, c^2$ **where *c* is the constant of integration.**	**1.5**

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Q.No	What to look for	Marks
	Compares the above equation with the general equation of parabola and concludes that the solution of the given differential equation represents the family of parabola.	**0.5**
8	**Rewrites the given differential equation as** $\frac{dy}{dx} = 3 - 2\frac{y}{x}$ **.** **Takes** $y = vx$ **and differentiates it to get** $\frac{dy}{dx} = v + x\frac{dv}{dx}$ **.**	**1**
	Rewrites the differential equation using the substitution in the above step as: $\frac{dv}{(1-v)} = 3\frac{dx}{x}$	**0.5**
	Integrates the above equation as: $-\log\|1 - v\| + \log\|C_1\| = 3\log\|x\|$ **where** C_1 **is the arbitrary constant.** **(Award full marks for equivalent appropriate answers.)**	**0.5**
	Simplifies the equation further and substitutes v **as** $\frac{y}{x}$ **to obtain the general solution of the given differential equation as:** $\left\|\frac{C_1}{1-v}\right\| = x^3$ $\Rightarrow \pm C_1 . \frac{x}{(x-y)} = x^3$ $\Rightarrow x^3 = \frac{x}{(x-y)}C, \text{ where } C = \pm C_1$ **(Award full marks for equivalent appropriate answers.)**	**0.5**
	Concludes that: $f(x, y) = \frac{x}{(x-y)}$	**0.5**
9	**Uses the variable separable form to rewrite the given equation as:** $\frac{dy}{y} = k\,dt$	**0.5**

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Q.No	What to look for	Marks
	Integrates the above equation on both sides to get: **ln y = kt + c, where c is a constant.**	**0.5**
	Writes the above equation in terms of e as: $y = e^{(kt + c)}$ $\Rightarrow y = e^{kt} \times e^{c}$	**1**
	Uses the given conditions to write: **At t = 0, $y = e^{c} = e^{10}$, where e^{10} is the initial population.**	**1**
	Uses the given condition to write: **At t = 5, $y = \frac{3}{4} e^{10} = e^{5k} \times e^{10}$** $\Rightarrow e^{5k} = \frac{3}{4}$	**1**
	Takes the natural logarithm on both sides to find the value of k as -0.058. The working may look as follows: **5k = ln 3 - ln 4** **=> 5k = 1.09 - 1.38** **=> 5 k = -0.29** $\Rightarrow k = \frac{-0.29}{5} = -0.058$	**1**
10	**i) Differentiates the given equation with respect to x as:** $2(y - c)\, y' = 3x^2$	**0.5**
	Simplifies the above differential equation to express c in terms of x, y and y' as: $c = y - \frac{3x^2}{2y'}$	**1**

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Q.No	What to look for	Marks
	Uses the expression obtained in the above step to eliminate *c* from the original equation as: $[y - (y - \frac{3x^2}{2y'})]^2 = x^3$ $\Rightarrow 9x^4 = 4x^3(y')^2$ $\Rightarrow 4x^3(y')^2 - 9x^4 = 0$	**1.5**
	ii) Writes the order of the above differential equation as 1.	**1**
	iii) Writes the degree of the above differential equation as 2. **(If the differential equation obtained by the student is incorrect but have identified the order and degree of the incorrect equation correctly, then award 2 marks.)**	**1**
11	**Uses the information given to find the rate of change of quantity of salt as follows:** $\frac{dQ_t}{dt} = 0 - 3000 \times \frac{Q_t}{20000 + 5000t - 3000t} = \frac{-3Q_t}{20 + 2t}$	**1**
12	**Separates the variables of the differential equation as follows:** $\frac{dQ_t}{dt} = \frac{-3Q_t}{20 + 2t}$ $\Rightarrow \frac{dQ_t}{-3Q_t} = \frac{dt}{20 + 2t}$	**1**
	Integrates both sides to obtain the following: $\int \frac{dQ_t}{-3Q_t} = \int \frac{dt}{20 + 2t}$ $\Rightarrow -\frac{1}{3} \ln Q_t = \frac{1}{2} \ln (10 + t) + C$ **where C is an arbitrary constant.**	**1**

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Q.No	What to look for	Marks
13	**Takes initial condition $Q_t(0) = 2000$ to obtain C as follows:** $-\frac{1}{3}\ln 2000 = \frac{1}{2}\ln 10 + C$ $C = -\frac{1}{3}\ln 2 - \frac{3}{2}\ln 10$	**1**
	Frames the equation as: $-\frac{1}{3}\ln Q_t = \frac{1}{2}\ln(10 + t) - \frac{1}{3}\ln 2 - \frac{3}{2}\ln 10$	**1**

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